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1984 J. Phys. A: Math. Gen. 17 3277

(http://iopscience.iop.org/0305-4470/17/16/027)

J. Phys. A: Math. Gen. 17 (1984) 3277-3282. Printed in Great Britain

Supergauge invariant Lagrangians from Noether identities

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Received 17 April 1984

Abstract. Starting from the supergauge invariance suggested by physical requirements, we set up the corresponding Noether identities and determine the most general Lagrangian which satisfies them.

1. Introduction

When a physical system is described by a Lagrangian with a certain number of gauge symmetries, then not all the dynamical coordinates are independent, since there exists exactly the same number of identities among the equations of motion. This is the content of the well known second Noether's theorem (Noether 1918), which applies both in the infinite-dimensional case (e.g. gauge field theories) as well as in the finite-dimensional one (e.g. constrained dynamics). The inverse problem arising from Noether's theorem can be formulated as the determination of the most general Lagrangian presenting a given number of gauge invariances (Sorace 1977, Gomis *et al* 1983). Although the solution of the inverse problem is considerably more difficult, this is, however, the most common phenomenological situation to be faced when looking for the description of a dynamical system whose gauge symmetries are suggested by physical requirements.

The purpose of this paper is to show how the inverse problem can be solved in the case of a finite-dimensional graded system invariant under reparametrisations of world lines and supergauge transformations, thus determining the most general Lagrangian admitting such invariances.

In order to give an account of the procedure to be followed, we briefly sketch the case of a single pointlike scalar particle, whose action is assumed to be invariant under the transformation of the evolution parameter

$$\tau \mapsto \bar{\tau} = \varphi(\tau) \tag{1.1}$$

where φ is a monotonic increasing function and equal to identity on the boundary of the τ-interval in which the evolution is considered, but otherwise arbitrary. If we assume a Lagrangian function only dependent on the particle position x^μ and its velocity $\dot{x}^\mu = dx^\mu/d\tau$, then the invariance under (1.1) immediately implies that the Lagrangian $L(x, \dot{x})$ has to be a homogeneous function of the first degree in $\dot{x}$.

An alternative description can be given in terms of a Lagrangian involving, in addition to x and $\dot{x}$, an 'einbein' e and its τ-derivative $\dot{e}$. We thus obtain a formalism which is valid also for massless particles and has an interesting interpretation in terms of one-dimensional gravity (Brink *et al* 1976, 1977). The invariance of the action under (1.1) now implies the invariance of the Lagrangian $L = L(x, \dot{x}, e, \dot{e})$ under the local variations

$$\bar{\delta}x^\mu = [\tau - \varphi(\tau)]\dot{x}^\mu, \qquad \bar{\delta}e = (\mathrm{d}/\mathrm{d}\tau)[(\tau - \varphi(\tau))e] \tag{1.2}$$

Denoting by $\delta/\delta q = \partial/\partial q - (\mathrm{d}/\mathrm{d}\tau)\,\partial/\partial\dot{q}$ the Euler–Lagrange operator, the Noether theorem yields the identity

$$\dot{x}^\mu \frac{\delta L}{\delta x^\mu} + \dot{e}\frac{\delta L}{\delta e} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(e\frac{\delta L}{\delta e}\right) \equiv \frac{\mathrm{d}}{\mathrm{d}\tau}\left(L - \dot{x}^\mu \frac{\partial L}{\partial \dot{x}^\mu} - \dot{e}\frac{\partial L}{\partial \dot{e}} - e\frac{\delta L}{\delta e}\right) = 0. \tag{1.3}$$

As (1.3) is an identity, the coefficients of the accelerations must separately vanish. We thus get the equations

$$\frac{\partial^2 L}{\partial \dot{x}^\mu\, \partial \dot{e}} = \frac{\partial^2 L}{\partial \dot{e}^2} = 0 \tag{1.4}$$

whose general solution, $L = F(x, e)\dot{e} + G(x, \dot{x}, e)$, can also be written in the form $L = H(x, \dot{x}, e) + (\mathrm{d}/\mathrm{d}\tau)K(x, e)$. Neglecting the total derivative and substituting $L = H$ into (1.3), we see that

$$\dot{x}^\mu \frac{\partial L}{\partial \dot{x}^\mu} + e\frac{\partial L}{\partial e} = L. \tag{1.5}$$

Therefore L is a homogeneous function of the first degree in $\dot{x}$ and e, a result independent of any other requirement of invariance that the system could satisfy. If, in particular, we look for a Poincaré scalar Lagrangian, we get

$$L = \sqrt{\dot{x}^2}\,h(\dot{x}^2/e^2) + eg(\dot{x}^2/e^2). \tag{1.6}$$

The choice $h(z) = \frac{1}{2}\sqrt{z}$, $g(z) = \frac{1}{2}m^2$ reproduces a known result, already present in the literature (Brink *et al* 1977).

In § 2 we introduce the invariance under supergauge transformations required for the description of a relativistic spinning particle, we set up the corresponding Noether identities and we determine the most general Lagrangian which satisfies them.

In § 3 we show how the previous results can be brought to bear on the problem of finding the general Lagrangian function for some two-particle systems, considering the consequences of assuming the invariance under independent reparametrisations of the particle world lines.

2. General Lagrangian for a relativistic spinning particle

In the framework of supergravity in one dimension, to describe a relativistic spinning particle we introduce, besides the usual coordinates x^μ, a set of Grassmann variables ξ^μ, ξ_5 connected to the spin, an 'einbein' e and an auxiliary odd variable ψ. The action of the particle is required to be invariant under the transformations (Brink *et al* 1977, Galvao and Teitelboim 1980, Sundermeyer 1982)

$$\begin{aligned}
\delta\tau &= \varphi(\tau) - \tau \equiv -a(\tau)\\
\bar{\delta}x^{\mu} &= a(\tau)\dot{x}^{\mu} + \mathrm{i}\alpha(\tau)\xi^{\mu}\\
\bar{\delta}e &= (\mathrm{d}/\mathrm{d}\tau)(a(\tau)e) + \mathrm{i}\alpha(\tau)\psi\\
\bar{\delta}\xi^{\mu} &= a(\tau)\dot{\xi}^{\mu} + (\alpha(\tau)/2e)(2\dot{x}^{\mu} - \mathrm{i}\psi\xi^{\mu})\\
\bar{\delta}\xi_5 &= a(\tau)\dot{\xi}_5 - m\alpha(\tau)\\
\bar{\delta}\psi &= (\mathrm{d}/\mathrm{d}\tau)(a(\tau)\psi + 2\alpha(\tau))
\end{aligned} \tag{2.1}$$

where $a(\tau)$ is the usual even gauge parameter associated with the reparametrisation, while $\alpha(\tau)$ is an odd gauge parameter related to supergauge transformations.

According to the second Noether theorem, we now have the two identities

$$\dot{x}^{\mu}\frac{\delta L}{\delta x^{\mu}} + \dot{\xi}^{\mu}\frac{\delta L}{\delta \xi^{\mu}} + \dot{\xi}_5\frac{\delta L}{\delta \xi_5} - e\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\delta L}{\delta e} - \psi\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\delta L}{\delta \psi} = 0 \tag{2.2}$$

$$\mathrm{i}\xi^{\mu}\frac{\delta L}{\delta x^{\mu}} + \frac{1}{2e}(2\dot{x}^{\mu} - \mathrm{i}\psi\xi^{\mu})\frac{\delta L}{\delta \xi^{\mu}} + \mathrm{i}\psi\frac{\delta L}{\delta e} - m\frac{\delta L}{\delta \xi_5} - 2\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\delta L}{\delta \psi} = 0 \tag{2.3}$$

whose general solution is to be investigated. The same procedure used in (1.3), when applied to (2.2), shows that the dependence of the Lagrangian on the variables $\dot{e}$ and $\dot{\psi}$ can be absorbed in a total derivative, so that $\dot{e}$ and $\dot{\psi}$ can be neglected. Moreover, as we want to consider a Poincaré invariant system, we shall also drop the dependence on x so that $L = L(\dot{x}, e, \xi, \dot{\xi}, \xi_5, \dot{\xi}_5, \psi)$.

As in § 1, (see (1.3)), the Noether identity (2.2) simply implies that L is a homogeneous function of the first degree in the variables $(\dot{x}, e, \dot{\xi}, \dot{\xi}_5, \psi)$, while a greater care is needed to find the consequences of the identity (2.3), related to supergauge transformations.

Again, as (2.3) must identically hold, the coefficients of the accelerations have to vanish separately. After some lengthy algebra, we see that (2.3) gives the conditions

$$T\frac{\partial L}{\partial \dot{x}^{\mu}} = T\frac{\partial L}{\partial \dot{\xi}^{\mu}} = T\frac{\partial L}{\partial \dot{\xi}_5} = T\frac{\partial L}{\partial \psi} = T\frac{\partial L}{\partial e} = 0 \tag{2.4}$$

$$\begin{aligned}
&\left(\mathrm{i}\dot{\xi}^{\mu}\frac{\partial}{\partial \dot{x}^{\mu}} - \frac{\mathrm{i}}{2e}\psi\dot{\xi}^{\mu}\frac{\partial}{\partial \dot{\xi}^{\mu}} + \frac{1}{2e}(2\dot{x}^{\mu} - \mathrm{i}\psi\xi^{\mu})\frac{\partial}{\partial \xi^{\mu}} - m\frac{\partial}{\partial \xi_5} + \mathrm{i}\psi\frac{\partial}{\partial e}\right)L\\
&\qquad = \left(\dot{\xi}^{\mu}\frac{\partial}{\partial \xi^{\mu}} + \dot{\xi}_5\frac{\partial}{\partial \xi_5}\right)TL
\end{aligned} \tag{2.5}$$

where T is the differential operator

$$T = \mathrm{i}\xi^{\mu}\frac{\partial}{\partial \dot{x}^{\mu}} + \frac{1}{2e}(2\dot{x}^{\mu} - \mathrm{i}\psi\xi^{\mu})\frac{\partial}{\partial \dot{\xi}^{\mu}} - m\frac{\partial}{\partial \dot{\xi}_5} + 2\frac{\partial}{\partial \psi}. \tag{2.6}$$

We now recast (2.4) in a form more suited for calculations, namely

$$\begin{aligned}
\frac{\partial}{\partial \dot{\xi}^{\mu}}TL &= \frac{\partial}{\partial \dot{\xi}_5}TL = \left(\frac{\partial}{\partial \psi} + \tfrac{1}{2}\mathrm{i}\xi^{\mu}\frac{\partial}{\partial \dot{x}^{\mu}}\right)TL\\
&= \left(\dot{x}^{\mu}\frac{\partial}{\partial \dot{x}^{\mu}} + e\frac{\partial}{\partial e} + \psi\frac{\partial}{\partial \psi}\right)TL = \frac{\partial}{\partial \dot{x}^{\mu}}TL - \frac{1}{e}\frac{\partial L}{\partial \dot{\xi}^{\mu}} = 0.
\end{aligned} \tag{2.7}$$

The first two equations say that TL is independent of $\dot{\xi}^\mu$ and $\dot{\xi}_5$. The third one shows that TL can depend on $\dot{x}^\mu$ only through the combination

$$z^\mu = \dot{x}^\mu - \tfrac{1}{2}\mathrm{i}\psi\xi^\mu. \tag{2.8}$$

The fourth equation establishes that TL is homogeneous of zero degree in $(\dot{x}^\mu, \psi, e)$ so that $TL = F(z^\mu, \xi^\mu, \xi_5, e)$, F being any homogeneous function of zero degree in z and e. Recalling that TL is an odd Poincaré invariant quantity, we can write

$$\begin{aligned} TL &= f\left(\frac{z^2}{e^2}\right)\xi_5 + g\left(\frac{z^2}{e^2}\right)\frac{(z\cdot\xi)}{e} \\ &= f\left(\frac{\dot{x}^2}{e^2}\right)\xi_5 - \mathrm{i}f'\left(\frac{\dot{x}^2}{e^2}\right)\psi\frac{(\dot{x}\cdot\xi)}{e^2}\xi_5 + g\left(\frac{\dot{x}^2}{e^2}\right)\frac{(\dot{x}\cdot\xi)}{e} \end{aligned} \tag{2.9}$$

where the prime denotes differentiation with respect to the argument. Using (2.9) and the last of equations (2.7), after some straightforward integrations and expanding the result on the Grassmann algebra of the odd variables, we get

$$\begin{aligned} L = {} & L_0 + L_1\xi_5\dot{\xi}_5 + L_2\xi_5\psi + L_3\xi_5(\dot{x}\cdot\xi) + L_4\dot{\xi}_5\psi + L_5\dot{\xi}_5(\dot{x}\cdot\xi) \\ & + L_6\psi(\dot{x}\cdot\xi) + L_7\xi_5\dot{\xi}_5\psi(\dot{x}\cdot\xi) + g(\dot{\xi}\cdot\xi) + 2f'\frac{(\dot{x}\cdot\dot{\xi})}{e}\xi_5 \\ & + 2g'\frac{(\dot{x}\cdot\dot{\xi})(\dot{x}\cdot\xi)}{e^2} - 2\mathrm{i}f''\frac{(\dot{x}\cdot\dot{\xi})\psi(\dot{x}\cdot\xi)\xi_5}{e^3} + \mathrm{i}f'\frac{(\dot{\xi}\cdot\xi)}{e}\psi\xi_5 \end{aligned} \tag{2.10}$$

where f, g are defined in (2.9) and $L_i = L_i(\dot{x}, e)$, $i = 0$–7. To obtain a consistent expression we apply the operator (2.6) to the Lagrangian (2.10) and compare the result with (2.9). Using also the equation (2.5) we finally get the following set of conditions

$$\begin{aligned} & L_0 = -\mathrm{i}m^2be - \mathrm{i}\frac{\dot{x}^2}{e}g - \mathrm{i}c, && L_1 = \frac{L_2}{m} = -\frac{1}{m}f\left(\frac{\dot{x}^2}{e}\right) = b. \\ & L_3 = L_4 = L_5 = L_7 = 0, && L_6 = -\frac{g}{2} - 2\frac{\dot{x}^2}{e^3}g', \end{aligned} \tag{2.11}$$

b and c being arbitrary constants.

We have thus found the general Lagrangian

$$\begin{aligned} L = {} & -\mathrm{i}m^2be - \mathrm{i}\frac{\dot{x}^2}{e}g + b\xi_5\dot{\xi}_5 + mb\xi_5\psi \\ & -\left(\frac{g}{e} + 2\frac{\dot{x}^2}{e^3}g'\right)\psi(\dot{x}\cdot\xi) + g(\dot{\xi}\cdot\xi) + 2\frac{g'}{e^2}(\dot{x}\cdot\dot{\xi})(\dot{x}\cdot\xi). \end{aligned} \tag{2.12}$$

Notice that for $b = \tfrac{1}{2}\mathrm{i}$ and $g = \text{constant} = \tfrac{1}{2}\mathrm{i}$ we get the Lagrangian given in (Brink *et al* 1977). Eliminating the einbein e we also recover the Lagrangian studied in Galvao and Teitelboim (1980).

3. Lagrangian functions for systems with two independent reparametrisations

Let us consider a system composed of two relativistic spinning particles described by the set of variables $(x_a^\mu, e_a, \xi_a^J, \psi_a) \equiv (q_a^A)$, $a = 1, 2$, where μ and J respectively denote the indices of the even and odd coordinates, (e_a, ψ_a) are the even/odd einbeins eventually necessary to describe the theory and finally the notation q_a^A is used whenever no distinction of the even/odd coordinates is needed. We shall use the convention of summing over the repeated μ, J and A indices but not on the particle index a.

If we require that the action of the system is invariant under independent reparametrisations of each world line, then the two particles are necessarily non-interacting. This result has been proved (Giachetti and Sorace 1979, Barducci *et al* 1984) using the integrability condition for vector fields on graded manifolds (Giachetti and Ricci 1984). We shall produce here a slightly different and simpler proof directly in terms of the dynamical coordinates.

The invariance under independent reparametrisations gives rise to the Noether identities

$$\dot{x}_a^\mu \frac{\delta L}{\delta x_a^\mu} + \dot{\xi}_a^J \frac{\delta L}{\delta \xi_a^J} - e_a \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\delta L}{\delta e_a} - \psi_a \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\delta L}{\delta \psi_a} = 0, \qquad a = 1, 2 \tag{3.1}$$

or, equivalently,

$$\dot{x}_a^\mu \frac{\delta L}{\delta x_a^\mu} + \dot{\xi}_a^J \frac{\delta L}{\delta \xi_a^J} + \dot{e}_a \frac{\delta L}{\delta e_a} + \dot{\psi}_a \frac{\delta L}{\delta \psi_a} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(e_a \frac{\delta L}{\delta e_a} + \psi_a \frac{\delta L}{\delta \psi_a}\right). \tag{3.2}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \sum_{a=1}^{2} \left(\ddot{q}_a^A \frac{\partial}{\partial \dot{q}_a^A} + \dot{q}_a^A \frac{\partial}{\partial q_a^A}\right) \tag{3.3}$$

equations (3.2) can be written in the form

$$\left(\ddot{q}_a^A \frac{\partial}{\partial \dot{q}_a^A} + \dot{q}_a^A \frac{\partial}{\partial q_a^A}\right) L = \frac{\mathrm{d}}{\mathrm{d}\tau} \sum_{b=1}^{2} \varepsilon_{ab} \left(L - \dot{q}_b^A \frac{\partial L}{\partial \dot{q}_b^A} - e_b \frac{\delta L}{\delta e_b} - \psi_b \frac{\delta L}{\delta \psi_b}\right) \tag{3.4}$$

where ε_{ab} denotes the skew-symmetric tensor in two dimensions. Equation (3.4) can be regarded as the statement that the two differential forms

$$\omega_a = \mathrm{d}\dot{q}_a^A \frac{\partial L}{\partial \dot{q}_a^A} + \mathrm{d}q_a^A \frac{\partial L}{\partial q_a^A} \tag{3.5}$$

are exact. The condition $\mathrm{d}\omega_a = 0$ gives then

$$\frac{\partial^2 L}{\partial \dot{q}_1^A \, \partial \dot{q}_2^B} = \frac{\partial^2 L}{\partial \dot{q}_1^A \, \partial q_2^B} = \frac{\partial^2 L}{\partial q_1^A \, \partial \dot{q}_2^B} = \frac{\partial^2 L}{\partial q_1^A \, \partial q_2^B} = 0 \tag{3.6}$$

which prove that L splits into the sum of two terms $L_1(q_1^A, \dot{q}_1^A)$ and $L_2(q_2^A, \dot{q}_2^A)$.

We conclude by observing that any other property we may require must be directly searched for in the sub-Lagrangians L_1 and L_2. In particular all the results of the previous sections apply.

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